第三章 原子的量子态：玻尔模型

3.1.The work function for cesium is 1.9eV.

(a) Determine the threshold frequency and threshold wavelength of the photoelectric effect of cesium.

(b) If one wants to obtain a photoelectron with energy of 1.5eV,what wavelength of light is required?

铯的逸出功为 1.9eV，试求：

（1）铯的光电效应阈频率及阈值波长；

（2）如果要得到能量为 1.5eV 的光电子，必须使用多少波长的光照射？

Solution:(a)A photoelectric current flows only when the frequency of the incident light exceeds a certain threshold frequency for the mental cesium.When the frequency of the incident light υ equals the threshold frequency of cesium υ_0 ,that is, $\upsilon=\upsilon_0$,the stopping potential $V_0=0$,no electron can escape from the mental surface,which means the kinetic energy of the electron $E_k=0$.

According to the equation: $E_k=h\upsilon-\phi$

When $E_k=0$,we can obtain the threshold frequency of cesium:

$$\boxed{\upsilon_0=\frac{\phi}{h}=\frac{\phi c}{hc}=\frac{1.9eV\times3.0\times10^8 m/s}{1.24nm\cdot keV}=4.6\times10^{14}Hz}$$

The threshold wavelength of cesium:

$$\boxed{\lambda_0=\frac{c}{\upsilon_0}=\frac{hc}{h\upsilon_0}=\frac{hc}{\phi}=\frac{1.24\times10^3 nm\cdot eV}{1.9eV}=6.5\times10^{-7}m}$$

(b)If the energy of a photoelectron is 1.5eV,the wavelength of light is:

$$\boxed{\lambda=\frac{c}{v}=\frac{hc}{hv}=\frac{hc}{E_k+\phi}=\frac{1.24\times10^3 nm\cdot eV}{(1.5+1.9)eV}=364.7nm}$$

3.3 What minimum kinetic energy must an electron have in order to allow an inelastic collision between the electron and a lithium ion Li^{2+} in its ground state to take place?

欲使电子与处于基态的锂离子 Li^{2+} 发生非弹性散射，电子至少具有多大的动能？

Solution:An inelastic collision is one in which the incoming electron collides with a lithium ion and excites a lithium ion in its ground state to a higher energy state.In order to calculate the minimum kinetic energy of an electron,we need to calculate the energy when a lithium ion jumps from ground state $n'=1$ (with energy $E_{n'}$) to the first excited state $n=2$ (with energy E_n):

$$E_n=-\frac{R_{Li^{2+}}hc}{n^2}\cdot Z^2$$

$$\Delta E_{12}=E_2-E_1=-\frac{Z^2R_{Li^{2+}}hc}{2^2}-\left(-\frac{Z^2R_{Li^{2+}}hc}{1^2}\right)=Z^2R_{Li^{2+}}hc\left(1-\frac{1}{2^2}\right)$$

$$=3^2\times109728.8cm^{-1}\times1.24\times10^3\,nm\cdot eV\times\frac{3}{4}$$

$$=3^2\times13.6\times\frac{3}{4}eV=91.8eV$$

Or

$$E_n=-\frac{1}{2}m_e(\alpha c)^2\frac{Z^2}{n^2}$$

$$\Delta E_{12}=E_2-E_1=-\frac{1}{2}m_e(\alpha c)^2Z^2\left(\frac{1}{2^2}-1\right)$$

$$=-\frac{1}{2}m_ec^2\alpha^2Z^2\cdot\left(-\frac{3}{4}\right)=-\frac{1}{2}\times0.511MeV\times\left(\frac{1}{137}\right)^2\times3^2\times\left(-\frac{3}{4}\right)$$

$$\approx-13.6\times3^2\times\left(-\frac{3}{4}\right)eV=91.8eV$$

3.5 (a)In the case of thermal equilibrium,the distribution of the atoms in different energy states is given by the Boltzmann distribution,namely,the number of atoms in an excited state with energy of E_n is

$$N_n=N_1\frac{g_n}{g_1}e^{-(E_n-E_1)/kT},$$

Where N_1 is the number of atoms in the state with energy E_1, k is the Boltzmann constant,and g_n and g_1 are the statistical weights (determined by how many different ways one can put the electrons in each of the two states with energies E_n and E_1)of the corresponding states.For hydrogen atoms at a pressure of 1atm and a temperature of 20℃, how large must the container be to let one atom be in the first excited state?Take the statistical weights of the hydrogen atoms in the ground state and in the first excited state to be $g_1 = 2$ and $g_2 = 8$,respectively.Remember from thermodynamics $PV = \gamma RT$ where $\gamma =$ number of atoms present/Avogadro's number= N / N_A.

原子在热平衡条件下处于不同能量状态的数目是按玻尔兹曼分布的，即处于能量为E_n的激发态的原子数为：

$$N_n = N_1 \frac{g_n}{g_1} e^{-(E_n - E_1)/kT},$$

式中N_1是能量为E_1状态的原子数，k为玻尔兹曼常数，g_n和g_1为相应能量状态的统计权重，试问：原子态的氢在一个大气压、20℃温度的条件下，容器必须多大才能有一个原子处在第一激发态？已知氢原子处于基态和第一激发态的统计权重分别为$g_1 = 2, g_2 = 8$

(b)Let electrons collide with hydrogen gas at room temperature.In order to observe the H_α line,what is the minimum kinetic energy of the electrons?

电子与室温下的氢原子气体相碰撞，要观察到H_α线，电子的最小动能为多大？

Solution:(a) In order to let one atom be in the first excited state(n=2), that is, $N_n = N_2 = 1$,according to the expression:

$$N_n = N_1 \frac{g_n}{g_1} e^{-(E_n - E_1)/kT},$$

We can obtain the number of atoms in the ground state:

$$N_1 = N_2 e^{(E_2 - E_1)/kT} \frac{g_1}{g_2}$$

Where,the energy for an electron of a hydrogen atom jumps from ground state to the first excited state is:

$$\Delta E_{12} = E_2 - E_1 = E_1\left(\frac{1}{2^2} - 1\right) = -\frac{3}{4} \times (-13.6)eV = 10.2eV$$

According to the equation:

$$PV_1 = \frac{N_1}{N_A}RT = N_1 kT$$

Hence,we obtain the volume of the container :

$$V_1 = \frac{N_1}{P}kT = \frac{N_2 e^{\Delta E_{12}/kT}\frac{g_1}{g_2}}{P}kT$$

Substituting the following data:

$$\begin{aligned}
&N_2 = 1\\
&k = 1.38\times 10^{-23} J/K\\
&T = 293K\\
&\frac{g_1}{g_2} = \frac{1}{4}\\
&\Delta E_{12} = 10.2eV = 1.634\times 10^{-18} J\\
&P = 1.01\times 10^5 Pa
\end{aligned}$$

The volume of the container is:

$$V = 2.6\times 10^{149} m^3$$

(b)In order to observe the H_α line,that is,the electron transits from n=3 to n=2,the energy to move an electron in the ground state of hydrogen to state n=3:

$$\boxed{\begin{aligned}
&E_n = -\frac{Rhc}{n^2} = \frac{E_1}{n^2}, or, E_n = -\frac{1}{2}m_e(\alpha c)^2\frac{1}{n^2} = \frac{E_1}{n^2}\\
&\Delta E_{13} = E_3 - E_1 = E_1\left(\frac{1}{3^2}-1\right) = -13.6eV\times(-\frac{8}{9}) = 12.09eV
\end{aligned}}$$

So the minimum kinetic energy of the electrons should equal 12.09eV.

3.6 In the range of wavelengths from 950 $\mathring{A}$ to 1250 $\mathring{A}$,what spectral lines are included in the absorption spectrum of a hydrogen atom?

在波长从950 $\mathring{A}$ 到1250 $\mathring{A}$ 的光带范围内，氢原子的吸收光谱中包含哪些谱线？

Solution:The energy to move an electron in the ground state of hydrogen atom $n' = 1$(with energy $E_{n'}$)to a higher state n (with energy E_n):

$$\Delta E_{1n} = E_n - E_1 = E_1\left(\frac{1}{n^2} - 1\right) = 13.6\left(1 - \frac{1}{n^2}\right)eV$$

According to the equation,the wavelength of a transition of energy E:

$$\boxed{\lambda = \frac{hc}{E} = \frac{1.24}{E} nm \cdot keV}$$

Where, $E = h\upsilon = E_n - E_1$ (an electron of hydrogen atom jumps from a higher state to the ground state,an electromagnetic wave of energy $h\upsilon$ would be emitted)

There is a correspondence between λ and E .For a given minimum λ ,there corresponds a definite maximum E ,that is,when $\lambda = 950\,\mathring{A}$,we can get the maximum excitation energy:

$$\boxed{E_{\max} = \frac{hc}{\lambda_{\min}} = \frac{1.24 nm \cdot keV}{950 \times 0.1 nm} = 13.052 eV}$$

Then,we can get the quantum number n:

$$\boxed{E_{\max} = 13.6\left(1 - \frac{1}{n^2}\right) \to n = 4.98}$$

Which means the electron can jump from n=4,n=3,n=2 to n=1 ,respectively.

①The wavelength for an electron jumps from n=4 to n=1,:

$$\boxed{\lambda_{41} = \frac{hc}{\Delta E_{14}} = \frac{hc}{E_4 - E_1} = \frac{hc}{E_1\left(\frac{1}{4^2} - 1\right)} = \frac{1.24 nm \cdot keV}{13.6\left(1 - \frac{1}{4^2}\right)eV} = 97.25 nm = 972.5\,\mathring{A}}$$

②The wavelength for an electron jumps from n=3 to n=1:

$$\boxed{\lambda_{31} = \frac{hc}{\Delta E_{13}} = \frac{1.24 nm \cdot keV}{13.6\left(1 - \frac{1}{3^2}\right)eV} = 102.57 nm = 1025.7\,\mathring{A}}$$

③The wavelength for an electron jumps from n=2 to n=1:

$$\boxed{\lambda_{21} = \frac{hc}{\Delta E_{12}} = \frac{1.24 nm \cdot keV}{13.6\left(1 - \frac{1}{2^2}\right)eV} = 121.57 nm = 1215.7\,\mathring{A}}$$

3.8 The photon emitted by a transition in ionized helium He^+ from its first excited state to its ground state can ionize a hydrogen atom in its ground state and make it emit an electron.Determine the velocity of the electron.

一次电离的氦离子 He^+ 从第一激发态向基态跃迁时所辐射的光子，能使处于基态的氢原子电离，从而放出电子，试求该电子的速度。

Solution:The energy for the photon emitted by a transition in ionized helium He^+ from its first excited state to its ground state:

$$\boxed{h\upsilon = E_2 - E_1 = E_1 Z^2\left(\frac{1}{2^2}-1\right) = -13.6\times 2^2\left(-\frac{3}{4}\right)eV = 40.8eV}$$

The energy to move an electron in the ground state of hydrogen to infinity is the ionization energy of hydrogen,which equals :

$$\boxed{E_\infty = \frac{1}{2}m(\alpha c)^2 = \frac{1}{2}\times 0.511\times 10^6\times\left(\frac{1}{137}\right)^2 = 13.6eV}$$

Hence,the kinetic energy of the electron is:

$$E_k = h\upsilon - E_\infty = (40.8-13.6)\times 1.602\times 10^{-19}J = 4.357\times 10^{-18}J$$

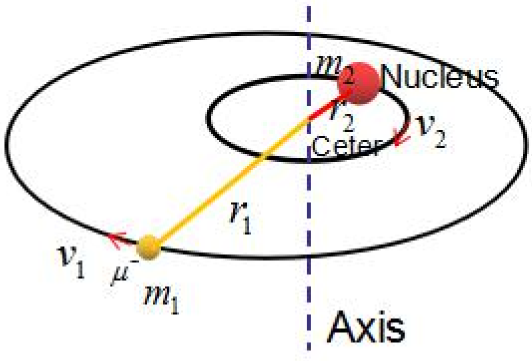
The velocity of the electron:

$$\boxed{v = \sqrt{\frac{2E_k}{m_e}} = \sqrt{\frac{2\times 4.357\times 10^{-18}J}{9.109\times 10^{-31}kg}} = 3.09\times 10^6 m/s}$$

3.10. A μ^- is an elementary particle just like the electron except its mass is 207 times as heavy as that of an electron.When the velocity of a μ^- is reasonably low,it will be captured by a proton and form a μ^- atom. Calculate:(a)The first Bohr radius of a μ^- atom.(b)The lowest energy of a μ^- atom.(c)The shortest wavelength of the Lyman series of a μ^- atom.

μ^- 子是一种基本粒子，除静止质量为电子质量的 207 倍外，其余性质与电子都一样，当它运动速度较慢时，被质子俘获形成 μ 子原子。试计算:(a) μ 子原子的第一玻尔轨道半径;(b) μ 子原子的最低能量;(c) μ 子原子莱曼线系中最短波长。

Solution:(a)According to the fig.Rotations of the μ^- and the nucleus in a hydrogen atom,we can get the following three equations:



$$\frac{m_1 v_1^2}{r_1} = \frac{e^2}{4\pi\varepsilon_0 r^2} \quad (1)$$

$$\frac{m_2 v_2^2}{r_2} = \frac{e^2}{4\pi\varepsilon_0 r^2} \quad (2)$$

$$m_1 v_1 r_1 + m_2 v_2 r_2 = n\hbar, n = 1, 2, 3\ldots \ (3)$$

Where,the reduced mass:

（原子核质量并不是无穷大，μ^-子绕核运动时，核不能固定不动，即为两体运动，原子核与μ^-子共同运动，电子质量要用折合质量代替）

$$\boxed{m_\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{207 m_e \times 1837 m_e}{(207 + 1837) m_e} = 186 m_e}$$

Orbit radius r_1, r_2:

$$r_1 = r \cdot \frac{m_2}{m_1 + m_2} = \frac{r m_\mu}{m_1}, (4)$$

$$r_2 = r \cdot \frac{m_1}{m_1 + m_2} = \frac{r m_\mu}{m_2}, (5)$$

两粒子绕同轴圆周运动，周期，角速度相同，两粒子绕轴运动的向心力也相同

（两粒子间的库仑力提供），由向心力公式

$$F = \frac{m 4\pi^2 r}{T^2} \to m_1 r_1 = m_2 r_2 \to m_1 r_1 = m_2 (r - r_1) \to r_1 = r \cdot \frac{m_2}{m_1 + m_2}$$

Substituting eqn(4),(5) into (1),(2),(3):

$$\frac{m_1 v_1^2}{r_1} = \frac{e^2}{4\pi\varepsilon_0 r^2} = \frac{m_1^2 v_1^2}{r m_\mu}, (6)$$

$$\frac{m_2 v_2^2}{r_2} = \frac{e^2}{4\pi\varepsilon_0 r^2} = \frac{m_2^2 v_2^2}{r m_\mu}, (7)$$

$$r m_\mu (v_1 + v_2) = n\hbar, (8)$$

$$\frac{eqn(6)}{eqn(7)} \to \frac{v_1}{v_2} = \frac{m_2}{m_1}, (9)$$

Substituting eqn(9)into(8):

$$m_\mu v_2 \left(\frac{m_2}{m_1} + 1\right) r = n\hbar \to m_2 v_2 r = n\hbar, (10)$$

Substituting eqn(10)into(7),we obtain the first Bohr radius of a μ^- atom :

$$\boxed{r = \frac{4\pi\varepsilon_0 r^2}{e^2} \frac{n^2\hbar^2}{r^2 m_\mu} = \frac{n^2}{186} \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2} = \frac{n^2}{186} \cdot a_1 = \frac{1}{186} \cdot 0.053nm = 0.00284 \mathring{A}}$$

或者直接套用电子轨道半径公式，但电子质量要替换成折合质量

$$\boxed{r_n = \frac{4\pi\varepsilon_0 \hbar^2}{m_\mu e^2} \cdot n^2 \to r = \frac{4\pi\varepsilon_0 \hbar^2}{186 m_e e^2} \cdot 1^2 = \frac{1}{186} a_1 = \frac{1}{186} \times 0.053nm = 0.00284 \mathring{A}}$$

(b).Energy can be expressed as the sum of kinetic energy and potential energy

$$E = E_k + E_P,$$

$$E_k = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} \cdot \frac{e^2 r_1}{4\pi\varepsilon_0 r^2} + \frac{1}{2} \cdot \frac{e^2 r_2}{4\pi\varepsilon_0 r^2} = \frac{1}{2} \frac{e^2}{4\pi\varepsilon_0 r},$$

$$E_p = -\frac{e^2}{4\pi\varepsilon_0 r},$$

$$\to E = -\frac{1}{2} \cdot \frac{e^2}{4\pi\varepsilon_0 r}, (11)$$

Substituting r into eqn(11),

$$E = -\frac{1}{2} \cdot \frac{e^2}{4\pi\varepsilon_0} \cdot \frac{186 m_e e^2}{4\pi\varepsilon_o n^2 \hbar^2} = -\frac{186 m_e e^4}{(4\pi\varepsilon_0)^2 \cdot 2\hbar^2 n^2}, (12)$$

The energy of the electrons in hydrogen:

$$E_n(H) = -\frac{m_e e^4}{(4\pi\varepsilon_0)^2 \cdot 2\hbar^2 n^2}, (13)$$

Hence the lowest energy of a μ^- atom:

$$\boxed{E = 186E_n(H) = 186\times(-13.6)eV = -2530eV}$$

或者直接用电子能量表达式，电子质量替换成折合质量：

$$\boxed{\begin{aligned} &E_n = -\frac{m_\mu e^4}{(4\pi\varepsilon_0)^2 \cdot 2\hbar^2 n^2} = -\frac{186 m_e e^4}{(4\pi\varepsilon_0)^2 \cdot 2\hbar^2 n^2} = 186E_n(H) \\ &\to E_1 = 186E_1(H) = 186\times(-13.6)eV = -2530eV \end{aligned}}$$

或

$$\boxed{\begin{aligned} &E_n = -\frac{1}{2} m_\mu (\alpha c)^2 \frac{1}{n^2} = -\frac{1}{2}\times 186 m_e (\alpha c)^2 \frac{1}{n^2} = 186E_n(H) \\ &\to E_1 = 186E_1(H) = 186\times(-13.6)eV = -2530eV \end{aligned}}$$

(C)According to the equation:

$$\lambda = \frac{hc}{E} = \frac{1.24}{E} nm\cdot keV$$

For a given maximum excitation energy,there correspond a minimum wavelength:

$$\boxed{\lambda_{\min} = \frac{1.24}{E_\infty - E_1} nm\cdot keV = \frac{1.24}{0-(-2530eV)} nm\cdot keV = 0.49nm}$$

3.11.The ratio of the Rydberg constant of hydrogen to that of heavy hydrogen is 0.999728 and the ratio of their nuclear masses is $m_H / m_D = 0.50020$. Calculate the ratio of the mass of a proton to the mass of an electron.

已知氢和重氢的里德伯常量之比为 0. 999728，而它们的核质量之比为 $m_H / m_D = 0.50020$，试计算质子质量与电子质量之比。

Solution:The Rydberg constant corresponding to the nucleus with mass A should be written as:

$$R_A = R\frac{1}{1+\frac{m_e}{m_A}}$$

Hence,we can write the Rydberg constant of hydrogen and heavy hydrogen,respectively.

$$R_H = R\frac{1}{1+\frac{m_e}{m_H}},$$

$$R_D = R\frac{1}{1+\frac{m_e}{m_D}},$$

$$\frac{R_H}{R_D} = \frac{1+\frac{m_e}{m_D}}{1+\frac{m_e}{m_H}} = \frac{1+\frac{m_e}{m_D}\cdot\frac{m_H}{m_H}}{1+\frac{m_e}{m_H}} = \frac{1+\frac{m_H}{m_D}\frac{m_e}{m_H}}{1+\frac{m_e}{m_H}} = \frac{1+0.50020\frac{m_e}{m_H}}{1+\frac{m_e}{m_H}} = 0.999728$$

$$\boxed{\rightarrow \frac{m_e}{m_H} = 0.5445\times 10^{-3}}$$

We obtain the ratio of the mass of a proton to the mass of an electron:

$$\boxed{\frac{m_H}{m_e} = \frac{1}{0.5445\times 10^{-3}} = 1836.5}$$